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# Attenuation and dispersion of sound in bubbly fluids via the Kramers–Kronig relations

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Sound propagation in a dilute bubble–liquid mixture is studied by means of the Kramers–Kronig relationships, which relate the real and imaginary parts of the general susceptibility of a linear medium. These relationships are adopted for the case of acoustic waves, where they become coupled integral equations. A simple but approximate procedure is used to obtain from these equations the phase speed of sound waves for the case when the attenuation coefficient is independently known. The procedure can be used to obtain the speed of propagation of sound waves in acoustic media having internal dissipation, but is here applied only to fluids containing radially pulsating bubbles. Approximate results for the speed of propagation and for the attenuation per wavelength are obtained for this case on the basis of a first-order estimate for the attenuation coefficient. These results are the same as those derived previously on the basis of model equations for bubbly liquids. They therefore provide additional support for those equations, while indicating some of their limitations.

## 1. Introduction

Sound propagation in bubbly liquids has been given considerable attention in the past as it plays a significant role in a variety of situations of interest. One reason that this area of research continues to be active is that bubbly clouds resulting from breaking waves in the ocean (Thorpe 1982) are believed to affect the propagation of sound waves used in underwater acoustic experiments.

The purpose of this work is to study the phase speed of sound waves in a bubbly medium without stipulating, *a priori*, dynamical models for the medium. The approach we use is based on the Kramers–Kronig (K–K) relationships. These relate the real and imaginary parts of the general susceptibility of a linear system, a quantity which connects the generalized displacement in dissipative systems to the generalized force applied to them (see, for example, Kittel 1958; Landau & Lifshitz 1958; Woods 1975; Pippard 1978; Beltzer 1988). As pointed out by Pippard, the K–K equations express, for these systems, a general truth that is independent of specific models. They are, therefore, ideal to test the suitability of given models for the response of a linear, dissipative system.

Several works exist where the K–K equations are applied in acoustic situations. These include those of O'Donnell, Jaynes & Miller (1981), who applied them to ultrasonic attenuation and phase velocity in systems which do not exhibit rapid frequency variations, that of Morfey & Howell (1980) who used their known form of a system containing a single degree of freedom to consider the effects of humidity on propagation, and those of Beltzer and Brauner (Brauner & Beltzer 1985; Beltzer &

Brauner 1987) who have applied them to the study of sound waves in composites. However, there seems to be no systematic application of those equations to the general problem of attenuation and dispersion of sound waves in fluids. In view of this, we first put the basic K–K equations in the acoustic context, and then introduce an approximate procedure to solve them for the phase speed assuming that the attenuation can be calculated independently, as is often the case in acoustics.

The procedure is then applied to a dilute suspension of equal-sized bubbles in a liquid, in the case when the bubbles pulsate radially under the effects of a plane sound wave. We show that the K–K approach yields results which reduce to those first derived by Kennard (1943), for the special case when the dissipation is due to acoustic radiation alone, and which agree with those results currently in use that were derived from more precise equations for a bubbly mixture (van Wijngaarden 1968, 1972; Plesset & Prosperetti 1977; Caflisch *et al.* 1985; Prosperetti 1986, 1987). This is not surprising, as considerable experimental data exist which appear to support the basic conclusions derived from these model equations away from resonance, particularly when the effects of bubble size distribution and frequency-dependence damping are taken into account as done by Commander & Prosperetti (1989). Nevertheless, the K–K approach also provides support to these models away from resonance, and shows that near resonance they are only approximately correct. Of course, nonlinear effects near resonance are probably more important.

## 2. The Kramers–Kronig equations in acoustics

The Kramers–Kronig equations are based on causality arguments regarding the response of a system to a given input. These arguments require that the generalized susceptibility of the system be a regular function of the frequency in the upper half of the complex domain (see, for example, Pippard 1978). The equations do not require any specific information about the system, other than assuming it to behave linearly, and connect, by means of some integral relations, the real and imaginary parts of the generalized susceptibility. Therefore, if either the real or the imaginary part of this generalized susceptibility is known, the unknown part may be obtained through integration in the frequency domain. Thus, for example, if one determines the real part through some experiment, the imaginary part can be obtained from the K–K equations, provided, of course, that the measurements cover those portions of the frequency range where the largest contributions to the integrals occur.

The generalized susceptibility may be defined as follows: Let the system be under the effect of a generalized force $F$. Then the generalized displacement of the system $X$ is given by $X = \chi F$, where $\chi$ represents the general susceptibility, a quantity which may be complex. Thus, $\chi = \chi' + \mathrm{i}\chi''$. If $\omega$ is used to represent the circular frequency, then the K–K equations may be expressed as

$$\chi'(\omega) - \chi'(\infty) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi''(\Omega)}{\Omega - \omega}\,\mathrm{d}\Omega, \tag{1}$$

$$\chi''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi'(\Omega) - \chi'(\infty)}{\Omega - \omega}\,\mathrm{d}\Omega, \tag{2}$$

where the principal part of the improper integrals is implied. Thus, given either $\chi'$ or $\chi''$, one of these equations can be used to obtain the other quantity.

Let us now put these relationships in a manner suitable for the calculation of acoustic attenuation and dispersion. First, we identify the acoustic quantity corresponding to the generalized susceptibility. An obvious choice is the acoustic admittance $u/p'$, where $u$ is the complex velocity of the system and $p'$ the acoustic pressure. This choice, however, requires an expression relating these two quantities and this is only available from model equations for the dynamic response of this system (see Appendix A). A less restrictive approach is to use the change of volume, or equivalently, the acoustic density $\rho'$, as the generalized displacement, and the applied acoustic pressure $p'$ as the generalized force. Thus, $\rho' = \chi p'$. However, we know from elementary theory that the acoustic pressure and density are related, in the linear approximation, by $p' = \rho' c^2$ where $c^2$ is the sound speed for the medium under consideration. Thus,

$$\frac{1}{c^2} = \chi' + \mathrm{i}\chi''. \tag{3}$$

This simply states that, in general, the quantity $c$ defined by (3) may be complex, implying, of course, that a phase lag, brought out by dissipation, exists between pressure and density. Instead of working with a complex speed, it is more convenient to work with a complex wavenumber $k$ defined by

$$k(\omega) = \frac{\omega}{c} = k_1 + \mathrm{i}k_2, \tag{4}$$

where $k_1$ and $k_2$ are real. Equating the real and imaginary parts of $k^2$ and of $\chi$ we obtain

$$\chi' = \frac{k_1^2 - k_2^2}{\omega^2}, \tag{5}$$

$$\chi'' = \frac{2k_1 k_2}{\omega^2}. \tag{6}$$

We now put these quantities in terms of attenuation and dispersion coefficients. For propagating monochromatic sound waves these are defined as follows. Consider a plane monochromatic wave propagating along the $x$-axis, and assume that its amplitude at some point $x_0$ is $A$. At points $x > x_0$, the acoustic variables are proportional to $A\exp\{-\alpha(x-x_0)+\mathrm{i}[k_1(x-x_0)-\omega t]\}$, where $\alpha$ is the *amplitude* attenuation coefficient. Thus,

$$\alpha(\omega) = k_2. \tag{7}$$

Similarly, the phase velocity of the waves is

$$c(\omega) = \frac{\omega}{k_1}, \tag{8}$$

where for convenience we have used the same symbol as in (3) to represent a real, frequency-dependent speed. In terms of $\alpha$ and $c$, $\chi'$ and $\chi''$ are given by

$$\chi' = \frac{1}{c^2(\omega)} - \frac{\bar{\alpha}^2}{c_f^2}, \tag{9}$$

$$\chi'' = \frac{2\bar{\alpha}}{c_f\, c(\omega)}, \tag{10}$$

where we have introduced a non-dimensional attenuation coefficient

$$\bar{\alpha} = \alpha(\omega)\frac{c_f}{\omega}, \tag{11}$$

and where $c_f$ denotes the phase speed in the fluid without dissipative mechanisms. Substituting these results in the original K–K equations, we obtain

$$\frac{c_f^2}{c^2(\omega)}-\bar{\alpha}^2=\frac{c_f^2}{c^2(\infty)}+\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{c_f}{c(\Omega)}\frac{\bar{\alpha}(\Omega)}{\Omega-\omega}\mathrm{d}\Omega \tag{12}$$

and

$$2\bar{\alpha}\frac{c_f}{c(\omega)}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{c_f^2/c^2(\Omega)-\bar{\alpha}^2-c_f^2/c^2(\infty)}{\Omega-\omega}\mathrm{d}\Omega, \tag{13}$$

where we have assumed that $\bar{\alpha}$ is zero at infinite frequencies.

It is also convenient to introduce the equilibrium, or zero-frequency speed of propagation, as this quantity can usually be obtained from general thermodynamic considerations. Thus, for $\omega=0$, (12) gives

$$\frac{c_f^2}{c^2(0)}=\frac{c_f^2}{c^2(\infty)}+\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{c_f}{c(\Omega)}\frac{\bar{\alpha}(\Omega)}{\Omega}\mathrm{d}\Omega, \tag{14}$$

where we have set $\bar{\alpha}(0)=0$. Substituting $c_f^2/c^2(\infty)$ from (13) into (11) yields

$$\frac{c_f^2}{c^2(\omega)}-\bar{\alpha}^2=\frac{c_f^2}{c^2(0)}+\frac{2\omega}{\pi}\int_{-\infty}^{\infty}\frac{c_f}{c(\Omega)}\frac{\bar{\alpha}(\Omega)}{\Omega(\Omega-\omega)}\mathrm{d}\Omega. \tag{15}$$

As they stand, (13) and (15) form a pair of coupled integral equations for $c(\omega)$ and $\bar{\alpha}$, respectively, and are not of much use in the acoustic case, even if one of these two quantities is known independently. Several methods exist that can be used to obtain approximate solutions from them. For our purposes, it is sufficient to make suitable assumptions about our unknowns when they appear under the integral sign. Thus, if we assume that the attenuation is very small, the changes of speed of propagation are then also small, and we can put $c_f/c(\Omega)$ equal to unity inside the integral in (15), and $\bar{\alpha}^2=0$ inside the integral in (13). Therefore, these equations give

$$\frac{c_f^2}{c^2(\omega)}-\bar{\alpha}^2=\frac{c_f^2}{c^2(0)}+\frac{2\omega}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\alpha}(\Omega)}{\Omega(\Omega-\omega)}\mathrm{d}\Omega, \tag{16}$$

$$2\bar{\alpha}\frac{c_f}{c(\omega)}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{c_f^2/c^2(\Omega)-c_f^2/c^2(\infty)}{\Omega-\omega}\mathrm{d}\Omega. \tag{17}$$

Clearly, if either $c_f/c(\omega)$ or $\bar{\alpha}$ is known, the second quantity can be obtained directly by integration of either (16) or (17). We consider here the case when a first-order approximation for $\bar{\alpha}$, say $\bar{\alpha}_0$, has been independently established, for example through experimental measurements. Then, the integral on the right-hand side of (16) can be evaluated so that (16) may be written as

$$\frac{c_f^2}{c^2(\omega)}-\bar{\alpha}^2=X(\omega), \tag{18}$$

where

$$X(\omega)=\frac{c_f^2}{c^2(0)}+\frac{2\omega}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\alpha}_0(\Omega)}{\Omega(\Omega-\omega)}\mathrm{d}\Omega. \tag{19}$$

Of course, if the attenuation is very small throughout the frequency range, we may also neglect $\bar{\alpha}^2$ in the left-hand side of (18), thus obtaining a first-order approximation for the speed ratio. When the attenuation is not small throughout, its square must be retained in (18). The second of the K–K equations can then be used to complete the system. Thus, substituting (18) into (13) yields

$$2\bar{\alpha}\frac{c_{\mathrm{f}}}{c(\omega)} = Y(\omega), \tag{20}$$

where

$$Y(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{X(\Omega)-X(\infty)}{\Omega-\omega}\,\mathrm{d}\Omega. \tag{21}$$

Equations (19) and (20) are two algebraic equations for the two unknowns, $\bar{\alpha}$ and $c_{\mathrm{f}}/c(\omega)$, which can be solved in terms of the known quantities $X$ and $Y$, giving

$$\frac{c_{\mathrm{f}}^2}{c^2(\omega)} = \tfrac{1}{2}X + \tfrac{1}{2}(X^2+Y^2)^{\frac{1}{2}} \tag{22}$$

and

$$\bar{\alpha} = \frac{\tfrac{1}{2}Y}{[\tfrac{1}{2}X + \tfrac{1}{2}(X^2+Y^2)^{\frac{1}{2}}]^{\frac{1}{2}}}. \tag{23}$$

Although approximate, this procedure is specially powerful in the acoustic case because the attenuation coefficient can often be independently calculated by using simple energy arguments in conjunction with the non-attenuated results predicted by acoustic theory. In the remaining sections of this work, we apply the method to study the propagation of sound waves in a fluid containing a dilute suspension of radially pulsating bubbles. The procedure, however, may be used for any acoustic situation where an attenuation coefficient is known.

## 3. Propagation in a dilute suspension of bubbles in a liquid

We consider the propagation of a monochromatic sound wave in a gas containing $n$ spherical bubbles per unit volume, all of the same radius $R_0$ and filled with a gas of density $\rho_{\mathrm{g}}$, which can pulsate radially in response to the acoustic waves. Because this radial motion involves dissipation, some energy is removed from the waves, so that they are attenuated as they travel in the bubbly liquid. This results in a non-zero *amplitude* attenuation coefficient which can be estimated from (Landau & Lifshitz 1959)

$$\alpha_0 = n\frac{\langle \dot{e}_{\mathrm{lost}}\rangle}{2c_{\mathrm{f}}\langle E_0\rangle}, \tag{24}$$

where $\dot{e}_{\mathrm{lost}}$ is the rate at which energy is dissipated per bubble, $c_{\mathrm{f}}$ is the ambient speed of sound in the fluid without bubbles, and $\langle E_0\rangle$ is the average acoustic energy per unit volume, computed by ignoring any dissipation effect which might exist. Thus, in the absence of these effects, the acoustic wave is described by a potential given by $\phi = A\exp[\mathrm{i}(k_0 x - \omega t)]$, where $A$ is real, and $k_0 = \omega/c_{\mathrm{f}}$. This potential gives an average energy density equal to $\langle E_0\rangle = \frac{1}{2}\rho_{\mathrm{f}} k_0^2 A^2$. Equation (24) assumes that the total dissipation rate is simply the sum of the rates due to each bubble; it is thus limited to dilute suspensions.

It remains to compute $\dot{e}_{\mathrm{lost}}$. This quantity can be computed by determining the rate at which the acoustic field must do work on the bubble to maintain its radial oscillations. Thus, if the radial velocity of points on the sphere surface is $U_{\mathrm{b}}$, and the radial force exerted by the acoustic field is $F_{\mathrm{a}}$, we have $\langle \dot{e}_{\mathrm{lost}}\rangle = \langle F_{\mathrm{a}} U_{\mathrm{b}}\rangle$. For bubbles small compared with the acoustic wavelength, the radial force is simply equal to the acoustic pressure evaluated at the mean location of the bubble's surface, times the surface area of the bubble. The pressure is simply $p' = \mathrm{i}\rho_f \phi$, with $\phi$ as given above. Evaluating the pressure at $x = 0$, where the bubble is located, we have

$$F_a = 4\pi R_0^2 \,\mathrm{i}\omega\rho_f A \exp(-\mathrm{i}\omega t).$$

Similarly, the bubble's velocity is

$$U_b = -\frac{\omega^2 A}{R_0} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega_0^2 - \omega^2 - 2\mathrm{i}\beta\omega}, \tag{25}$$

where $\omega_0$ is the resonant frequency of a bubble, as given by Minnaert's formula (Minnaert 1933)

$$\omega_0 = \frac{c_f}{R_0}(3\rho_g/\rho_f)^{\frac{1}{2}} = \tau_0^{-1}, \tag{26}$$

and where $\beta$ is the bubble's damping coefficient, which includes all dissipative effects. That is, $\beta$ is assumed to contain contributions from all dissipative processes, such as viscous and thermal effects, acoustic radiation, etc. For the purposes of this work we consider $\beta$ as a given constant, although each of its known separate contributions depends on frequency (Devin 1959; Eller 1970; Prosperetti 1977). In (26), we have introduced a timescale, $\tau_0$, which may be regarded as the relaxation time for the radial pulsations.

Using these results for $U_b$ and $F_a$, we obtain

$$\langle \dot{e}_{\text{lost}} \rangle = 2\pi R_0 A^2 \omega^4 \rho_f \frac{2\beta}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}. \tag{27}$$

We now substitute these results into our working equation for $\bar{\alpha}_0$ and find

$$\overline{\alpha_0} = \tfrac{1}{2} C_v N^2 \frac{b\omega\tau_0}{(1-\omega^2\tau_0^2)^2 + (b\omega\tau_0)^2}, \tag{28}$$

where $b = 2\beta\tau_0$ is a non-dimensional damping coefficient, $C_v$ is the volume concentration of the bubbles for dilute suspensions, given by

$$C_v = \tfrac{4}{3}\pi n R_0^3. \tag{29}$$

In (28) we have put

$$N^2 = \frac{\rho_f c_f^2}{\rho_g c_g^2}. \tag{30}$$

This quantity gives the ratio of the isentropic compressibility of the gas in the bubbles to that of the fluid around them. For an air–water mixture, $N^2$ is larger than $1.6 \times 10^4$. Therefore, the quantity $C_v N^2$ is not negligible, even for very small concentrations. Further, the non-dimensional damping constant $b$ is usually small. Thus, unless the volume fraction is very small, the above results predicts attenuation rates at resonance which are not generally small.

We now use this attenuation estimate in (16) to obtain

$$\frac{c_f^2}{c^2(\omega)} - \bar{\alpha}^2 = \frac{c_f^2}{c^2(0)} + C_v N^2 \frac{\omega\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{b\,\mathrm{d}x}{[(1-x^2)^2 + (bx)^2](x - \omega\tau_0)}. \tag{31}$$

For a dilute suspension, the first term in the right-hand side is given by

$$\frac{c_f^2}{c^2(0)} = 1 + C_v N^2. \tag{32}$$

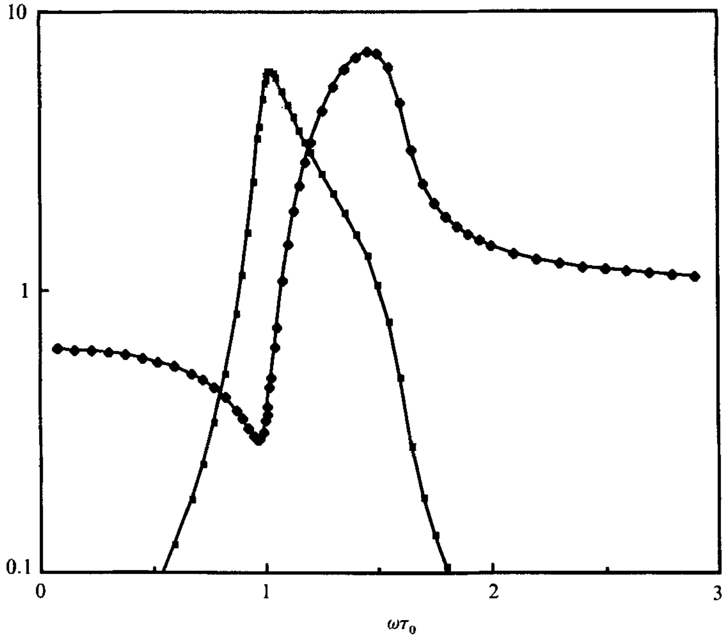


FIGURE 1. Attenuation $\bar{\alpha}$, and speed ratio $c^2(\omega)/c_f^2$ for $C_v = 10^{-4}$ and $b = 0.1$: ◆, $c_f^2/c^2(\omega)$; ■, $\bar{\alpha}$.

Next, we assume that $b$ is a constant, so that we can write

$$\frac{c_f^2}{c^2(\omega)} - \bar{\alpha}^2 = 1 + C_v N^2[1 + b\omega\tau_0 I], \tag{33}$$

where
$$I = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{[(1-x^2)^2 + (bx)^2]\,(x-\omega\tau_0)}. \tag{34}$$

This integral is evaluated in Appendix B, for small $b$, where it is shown that

$$I = \frac{1}{b}\omega\tau_0 \frac{1-\omega^2\tau_0^2 - b^2}{[(1-\omega^2\tau_0^2)^2 + (b\omega\tau_0)^2]}. \tag{35}$$

Substituting this into (33) we find

$$\frac{c_f^2}{c^2(\omega)} - \bar{\alpha}^2 = 1 + C_v N^2 \frac{1-\omega^2\tau_0^2}{[(1-\omega^2\tau_0^2)^2 + (b\omega\tau_0)^2]}. \tag{36}$$

The right-hand side of this equation gives the value of the quantity $X(\omega)$ defined in (18). For $Y(\omega)$ we have, on using the above equation as well as (19),

$$Y(\omega) = -C_v N^2 \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-x^2}{(1-x^2)^2 + (bx)^2}\frac{\mathrm{d}x}{x-\omega\tau_0}. \tag{37}$$

This yields, upon integration,

$$Y(\omega) = C_v N^2 \frac{b\omega\tau_0}{(1-\omega^2\tau_0^2)^2 + (b\omega\tau_0)^2}. \tag{38}$$

This completes the solution; the K–K values for the attenuation and dispersion are given by (22) and (23), with the quantities $X$ and $Y$ as given by (36) and (38), and figure 1 shows these results for $C_v = 10^{-4}$ and $b = 0.1$. Thus, in addition to providing

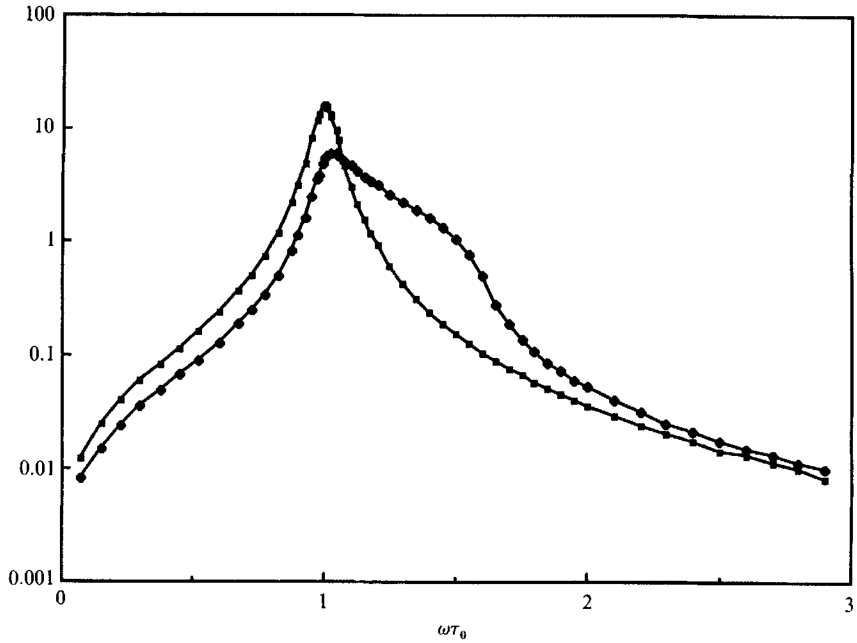


FIGURE 2. Attenuation coefficients for $C_v = 10^{-4}$ and $b = 0.1$: ■, $\overline{\alpha_0}$; ◆, $\bar{\alpha}$.

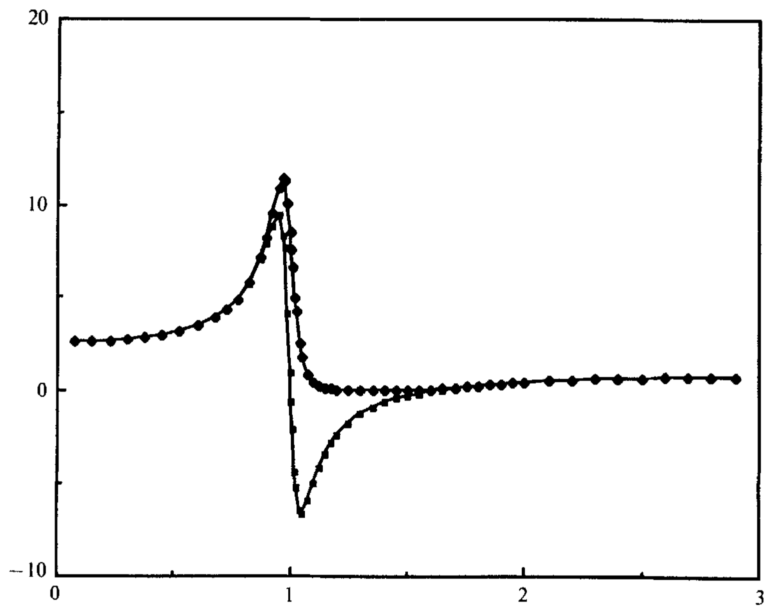


FIGURE 3. Inverse speed ratio $c_f^2/c^2(\omega)$ for $C_v = 10^{-4}$ and $b = 0.1$: ■, equation (A 3); ◆, equation (22) with (36) and (38).

a result for the speed of propagation in the bubbly liquid, the K–K equations result in an attenuation coefficient which differs, in the region near resonance, from the acoustic estimate given by (28), as shown in figure 2. Similarly, figure 3 compares our more complete result for the speed ratio, given by (22) together with (36) and (38), with that obtained from (18) by neglecting $\bar{\alpha}^2$ on the left-hand side of that equation.

Thus, except near resonance, the two sets of results are equivalent. Near resonance, however, the simpler equation for the speed ratio is incorrect, as it predicts negative values for the speed ratio in a band of frequencies around $\omega\tau_0 = 1$, whose width depends on $C_v$ and $b$. Incidentally, this incorrect result corresponds to that given by equation (A 3), which is derived in Appendix A from a simpler K–K system.

We now compare our results to those previously known. First, we note that they are basically the same result as those presented some time ago by Kennard (1943), except that the dissipation mechanism considered by him was due to acoustic radiation alone, that is, with $b\omega\tau_0 = (\sqrt{3}/N)(\omega\tau_0)^3$. To compare our solution to other, more recent, theories we first write our results as

$$\left(\frac{kc_f}{\omega}\right)^2 = 1 + C_v N^2 \frac{1}{1-\omega^2\tau_0^2 - ib\omega\tau_0}. \tag{39}$$

This result is equal to that derived by some investigators on the basis of a compressibility model for the bubbly liquid (see, for example, Meyer & Skudryzyk 1953; Fox, Curley & Larson 1955; Silberman 1957; Clay & Medwin 1977). It is also basically the same result as that derived by Wijngaarden (1972), Drumheller & Bedford (1979), Caflisch *et al.* (1985), and Prosperetti (1986, 1987), among others, on the basis of model equations for the fluid dynamic behaviour of a bubbly mixture. Therefore, the K–K approach, which does not rely on specific models lends some support to them, at least in the case of dilute suspensions of bubbles in liquids.

However, the agreement between our results and those currently in use shows that both are only approximate. One reason is that our final results are only an approximation to the K–K equations, at least in the region near resonance, where the attenuation and dispersion vary considerably. A second limitation relates to the non-dimensional damping coefficient $b = 2\beta\tau_0$, which was taken to be a constant in our formulation. As pointed out earlier, however, every damping mechanism is known to depend on frequency, but, provided that the variations with frequency of $\beta$ are not too abrupt, a frequency-dependent value of $2\beta\tau_0$ for $b$ will not modify the final results. This is shown by our earlier discussion of the damping coefficient due to radiation. There we substituted $(\sqrt{3}/N)(\omega\tau_0)^2$ for $b$ and obtained the correct result due to Kennard, even though this value of $b$ varies with frequency. The reason that enabled us to do this simple substitution was, of course, the sharp peak near $\omega\tau_0 = 1$ in the integrand in (34), which occurs when $b \ll 1$. In such cases, the variations of $b$ with frequency are of no consequence in determining the value of that integral, and therefore the value of $c_f/c(\omega)$. Only the value of $b$ at $\omega\tau_0 = 1$ is then relevant. For larger values of $b$, of course, the situation changes, and we can no longer merely substitute in the final results the values found for $b$ in damping studies. Rather, one must then integrate the K–K relationships using the actual dependence of $\beta$.

Part of this work was performed in the Acoustics System Branch of the Naval Research Laboratory. I am grateful to Drs Erskine and Pitre of that Branch for their interest in it. I am also grateful to the referees for their valuable comments. Appendix A, dealing with a simpler K–K system, is based on a suggestion by one of the referees.

## Appendix A. Sound waves in fluids with small dissipation

We consider here the propagation of plane waves in fluids having small amounts of internal dissipation. Examples could include pure fluids with internal degrees of freedom, as well as dilute particle–fluid mixtures. As we will see, the results obtained could also be applied to bubbly liquids away from resonance. Now, when dissipation is small, we may, in the acoustic case where all motions have a small amplitude, write the momentum-conservation equation for the medium under consideration as $\rho(\partial u/\partial t)+\partial p/\partial x \approx 0$. It is of course assumed that $\rho$, $u$ and $p$ can be suitably defined. Now, for plane monochromatic waves, this gives $u/p' = k/\omega\rho$. Putting $k = k_1+\mathrm{i}\alpha$, this relationship defines the general susceptibility as

$$\chi = (1/\rho)\,(1/c(\omega)+\mathrm{i}\bar{\alpha}).$$

Substituting this into (1) and (2) yields the considerably simpler K–K system

$$\frac{c_{\mathrm{f}}}{c(\omega)} = \frac{c_{\mathrm{f}}}{c(0)}+\frac{\omega}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\alpha}(\Omega)}{\Omega(\Omega-\omega)}\,\mathrm{d}\Omega \tag{A 1}$$

and

$$\bar{\alpha} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{c_{\mathrm{f}}/c(\Omega)-c_{\mathrm{f}}/c(\infty)}{(\Omega-\omega)}\,\mathrm{d}\Omega. \tag{A 2}$$

For small attenuation throughout the frequency range, these equations provide valid and useful approximations to the K–K equations, as may be seen by direct comparison with (16). In the case of strong dissipation, as with resonant bubbles in a liquid, they do not give accurate results in the region near resonance. Thus, in that instance, these equations give

$$\frac{c_{\mathrm{f}}}{c(\omega)} = 1+\tfrac{1}{2}C_v N^2\frac{1-\omega^2\tau_0^2}{[(1-\omega^2\tau_0^2)^2+(b\omega\tau_0)^2]}. \tag{A 3}$$

For some values of $C_v$ and $b$ this predicts negative values of the speed ratio.

## Appendix B. Evaluation of integral in equation (34)

We consider the integral (34) in the complex plane $z = x+\mathrm{i}y$. For the contour of integration, we take the real axis with a semicircular indentation of small radius around $\omega\tau_0$, and a semicircle of large radius connecting, in the limit as the radius goes to infinity, $x = +\infty$ with $x = -\infty$. The integral may be written as follows:

$$I = \mathrm{i}\,\mathrm{Res}\{2g(z_{\mathrm{u}1})+2g(z_{\mathrm{u}2})+g(\omega\tau_0)\}, \tag{B 1}$$

where

$$g(z) = \frac{1}{[(1-z^2)^2+(bz)^2]\,(z-\omega\tau_0)} = \frac{1}{(z-z_{\mathrm{u}1})\,(z-z_{\mathrm{u}2})\,(z-z_{11})\,(z-z_{12})\,(z-\omega\tau_0)}.$$

Here $z_{\mathrm{u}1}$ and $z_{\mathrm{u}2}$ are the poles of $g(z)$ in the upper plane, $z_{11}$ and $z_{12}$ are those in the lower plane. These quantities are given by

$$z_{\mathrm{u}1} = (1-b^2/2+\mathrm{i}b(1-b^2/4)^{\frac{1}{2}})^{\frac{1}{2}};\quad z_{\mathrm{u}2} = (1-b^2/2-\mathrm{i}b(1-b^2/4)^{\frac{1}{2}})^{\frac{1}{2}}.$$

In addition, we have, $z_{11} = -z_{\mathrm{u}1}$ and $z_{12} = -z_{\mathrm{u}2}$. Several relationships exist between these quantities which enable us to evaluate (B 1). Thus,

$$z_{\mathrm{u}1}\,z_{\mathrm{u}2} = 1;\quad z_{\mathrm{u}1}\,z_{\mathrm{u}2} = -1;\quad z_{12}\,z_{\mathrm{u}2} = -z_{\mathrm{u}2}^2;\quad z_{\mathrm{u}2}-z_{11} = z_{\mathrm{u}1}-z_{12}.$$

These give, after some algebra,

$$\pi I = \frac{1}{2(z_{\mathrm{u}1}+z_{\mathrm{u}2})}\frac{\omega\tau_0-(z_{\mathrm{u}1}+z_{\mathrm{u}2})}{(1-\omega^2\tau_0^2)^2+\omega\tau_0(z_{\mathrm{u}1}+z_{\mathrm{u}2})}+\frac{\pi\mathrm{i}}{(1-\omega^2\tau_0^2)^2+(b\omega\tau_0)^2}.$$

This result is still exact. To reduce it further, we take advantage of the assumed smallness of $b$ to expand $z_{u1}$ and $z_{u2}$, and obtain $z_{u1}+z_{u2} = ib+O(b^3)$. Thus,

$$\pi i = \frac{\pi/b}{(1-\omega^2\tau_0^2)^2+(b\omega\tau_0)^2}\{(\omega\tau_0-ib)[(1-\omega^2\tau_0^2)-ib\omega\tau_0]+ib\}$$

from which, (35) follows.

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